TP-95-001

January 1995

Interaction with a pre and post selected environment and recoherence

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Abstract

The interaction of an open system $\mathcal{S}$ with a pre- and post-selected environment is studied. In general, under such circumstances $\mathcal{S}$ can not be described in terms of a density matrix, *even when $\mathcal{S}$ in not post-selected*. However, a simple description in terms of a two-state (TS) is always available. The two-state of $\mathcal{S}$ evolves in ti me from an initially 'pure' TS to a 'mixed' TS and back to a final 'pure' TS. This generic process is governed by a modified Liouville equation, which is derived. For a sub-class of observables, which can still be described by an ordinary density matrix, this evolution generates recoherence to a final pure state. In some cases post-selection can even suppress any decoherence.

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1 Introduction

The interaction of an open quantum system with an environment [1] is traditionally analyzed while assuming a given, not necessarily known, initial state of the total closed system. In this case the open system can be described by a reduced density matrix which is obtained by tracing over the unknown environment's degrees of freedom. In this work we investigate circumstances in which the environment, and possibly also the open system, are bound to satisfy, not only an initial condition, but also a second final condition. In other words, we shall consider the interaction of a pre- and post selected environment with an open system.[2] Although, under usual circumstances, such a post selection is not realized, it is in principle not forbidden. Quantum Mechanics is (dynamically) time symmetric, and it is possible to conceive situations in which the initial and final conditions are selected according to some 'dynamical principle' (e.g. [3]).

We shall show that when the environment is post selected, the system can not generally be described in terms of a reduced density matrix. At any intermediate time, between the pre- and post-selection, there exists no pure or mixed state, which yields the correct probabilities for measurements in the open system, *even when the open system is not post-selected*. We suggest that is such cases, it is preferable, both practically and conceptually, to describe the open system by a new object which is a generalization of the density matrix.

It was recently suggested, that a quantum system should basically be described by an extension of the ordinary quantum state (or density matrix) called a "two-state" (TS), which is determined by two, initial and final, conditions [4, 5]. In the following we apply the formalism developed i n Ref. [5] to this problem. The probabilities for any measurement in the open system are shown to be derived from a reduced TS, i.e. the TS obtained by tracing over the environment's degrees of freedom. When the initial and final state of the environment are given by pure states, this reduced TS evolves in time from an initial 'pure

TS' to a a 'mixed TS' (of entangled form) at intermediate times, and finally back to a pure TS. Therefore, the effect of post-selecting the environment is to "recohere the TS". This process is dynamically expressed by a modified Liouville equation. As we shall show, the coefficients of the new terms in the this equation are time dependent, and tuned in such a way that the TS finally "recoheres".

It is well known, that interaction with an environment often causes decoherence in the open system. (For example see: [6, 7, 8, 9, 10]). In our case of post-selection, although the description in terms of a (pure or mixed) density matrix is generally invalidated, one can still find an effective density matrix for a limited class of observables. We show that the post-selection causes this effective density matrix to recohere to a final pure state [11]. In some cases, depending on the nature of the interaction, post-selection can suppress any decoherence.

This article proceeds as follows. In the next section we review shortly the two-state formalism of quantum mechanics and elaborate on some relevant details. In Section 3. we apply this formalism to the case of a pre and post selected system. A simple solvable example is given in Section 4. The modified Liouville equation which is satisfied by the two-state is derived in the last section using a perturbative approximation scheme and is applied to some cases. In the following we set $\hbar = 1$.

2 Quantum mechanics in terms of two-states

Two-states are particularly suitable in situations with two or more condition on a single quantum system. We now briefly review this formalism following Reference [5], and elaborate further on some relevant issues.

Consider a system $\mathcal{S}$ with a given Hamiltonian H_s. Let us assume that at t_1 and t_2 a complete set of measurements determine the states of $\mathcal{S}$ to be $|\psi_{in}(t_1)\rangle$ and $|\psi_{out}(t_2)\rangle$,

respectively. Now consider an ensemble of such identical systems which is defined by the latter two conditions. We are interested in probability distributions of observables that are measured in some intermediate time $t_2 > t > t_1$. The peculiarity of such a situation is that in general (as we shall see) these probabilities can not be derived from a single wave function or density matrix. It was therefore, suggested that the "state" of $\mathcal{S}$ at intermediate times should be described by a generalization of the ordinary wave function, which we call a 'two-state'. Generically, a TS, which we denote by $\hat{\varrho}$, is a non-Hermitian operator with the form:

$$\hat{\varrho} \;=\; |...\rangle\langle...| \tag{1}$$

At the left and right slots of $\hat{\varrho}$ one inserts the information due to the conditions at the t_1 and t_2 respectively. In the case of a closed system $\mathcal{S}$ we have:

$$\hat{\varrho}(t) = U(t-t_1)|\psi_{in}\rangle\langle\psi_{out}|U^\dagger(t_2-t) = |\psi_{in}(t)\rangle\langle\psi_{out}(t)| \tag{2}$$

where $U(t)$ is the unitary evolution operator.

More generally, two-states are elements of a Hilbert space $\mathcal{H}_{II}$, which is defined as follows. Given by a Hilbert space of states $\mathcal{H}_I = \{|\alpha\rangle\}$, we can construct the linear space $\mathcal{H}_{II} = \{|\alpha\rangle\langle\beta|\}$, where $|\alpha\rangle$ and $|\beta\rangle$ are any two elements of $\mathcal{H}_I$. The space $\mathcal{H}_{II}$ is a Hilbert space under the inner product:

$$\langle\hat{\varrho}_1, \hat{\varrho}_2\rangle \;\equiv\; \mathrm{tr}(\hat{\varrho}_1^\dagger\ \hat{\varrho}_2) \tag{3}$$

where the trace is over a complete set of states in $\mathcal{H}_I$. Mathematically, a TS, $\hat{\varrho} \in \mathcal{H}_{II}$, can always be expended in terms of a basis $\hat{\varrho}_{\alpha\beta} = |\alpha\rangle\langle\beta|$ of $\mathcal{H}_{II}$ as

$$\hat{\varrho} = \sum C_{\alpha\beta}\hat{\varrho}_{\alpha\beta} \tag{4}$$

A general $\hat{\varrho} \in H_{II}$ may not be reducible to the "generic form" (1). A non-generic TS with the "entangled" form (4) describes situations of a non-complete specification of the

conditions, that is, the final and/or initial conditions correspond to an entangled state of $\mathcal{S}$ with some other system, say $\mathcal{S}'$, whose degrees of freedom are traced out. In this case, we have two density matrix ρ_{in} and ρ_{out}, rather then two pure states as conditions. The conditions can be expressed as $\hat{\varrho}\hat{\varrho}^\dagger|_{t=t_1} = \rho_{in}(t_1)$ and $\hat{\varrho}^\dagger\hat{\varrho}|_{t=t_2} = \rho_{out}(t_2)$. In such circum stances, the occurrence of an entangled (non-generic) TS is due to the interaction of $\mathcal{S}$ and $\mathcal{S}'$ via the measurement device mediator, which is used to determine the conditions. Hence the dynamical evolution of the system is not modified (generic TS do not evolve in time to non-generic or vice versa). The TS of a closed system satisfies the Liouville equation:

$$i\partial_t\hat{\varrho} = [H, \hat{\varrho}] \tag{5}$$

In the following we shall study the appearance of entangled two-states (4) in a dynamical way through the interaction of $\mathcal{S}$ and $\mathcal{S}'$. To accommodate for this extra interaction we will need to modify the Liouville equation (5).

Given by a two-state $\hat{\varrho}(t)$ that corresponds to a pre and post selected ensemble, we can calculate the quantum mechanical probabilities for the result of any measurement at time t as follows. Let A be a Hermitian operator with a spectral expansion , $A = \sum aP_a$ in terms of projection operators $P_a = |a\rangle\langle a|$. Then, the probability to find $A = a$ is given by

$$\mathcal{P}rob(a;t) = \frac{|\langle P_a, \hat{\varrho}(t)\rangle|^2}{\sum_{a'}|\langle P_{a'}, \hat{\varrho}(t)\rangle|^2} \tag{6}$$

Therefore, in analogy with the ordinary expression for probability, the projection of $\hat{\varrho}$ on P_a, $\langle P_a, \hat{\varrho}\rangle$, can be interpreted as the TS amplitude. The absolute square of this amplitude is proportional to the probability. In general, this probability distribution can not be reduced to an expression in terms of a pure or mixed density matrix. To see this, notice that Equation (6) can also be written as

$$\mathcal{P}rob(a;t) = \frac{\langle\hat{\varrho}P_a, P_a\hat{\varrho}\rangle}{\sum_{a'}\langle\hat{\varrho}P_{a'}, P_{a'}\hat{\varrho}\rangle} = \frac{\mathrm{tr}P_a\rho(a)}{\sum_{a'}\mathrm{tr}P_{a'}\rho(a')} \tag{7}$$

where, $\rho(a) \equiv \hat{\varrho} P_a \hat{\varrho}^\dagger$. Therefore, this probability can be expressed in terms of a density matrix, only when $\rho(a)$ is independent of a.

Finally, we note that if the ensemble is only pre (or post) selected, the ordinary expression for the probability can be obtained as follows. Assuming that the final (unknown) measurement of some Hermitian operator $\hat{K}$ determines one of the eigenstates ψ_k, the probability to find $A = a$ is given by,

$$\mathcal{P}rob_I(a;t) = \sum_{\psi_k} \mathcal{P}rob(a) \mathcal{P}rob(\psi_k|\psi_{in}) = |\langle a|\psi_{in}\rangle|^2 \tag{8}$$

i.e. by the ordinary expression. In terms of the TS this yields

$$\mathcal{P}rob_I(a;t) = \frac{\mathrm{tr} P_a \rho_{in}(t)}{\mathrm{tr} \rho_{in}(t)} = \frac{\langle \hat{\varrho}(t), P_a \hat{\varrho}(t)\rangle}{\langle \hat{\varrho}(t), \hat{\varrho}(t)\rangle} \tag{9}$$

where $\rho_{in} = \hat{\varrho}\hat{\varrho}^\dagger$. This expression is to be compared with (7). Contrary to the former case of a pre- and post-selection, the latter expression depends only in the initial condition.

3 A system with a pre and post selected environment

Consider a closed system $\mathcal{S}_T$ which is composed of the sub-systems $\mathcal{S}$ and $\mathcal{S}_e$. Let the part $\mathcal{S}_e$ play the role of an environment $\mathcal{E}$. The Hamiltonian of the total system is

$$H_{tot} = H_s + H_e + H_{int} = H_0 + H_{int} \tag{10}$$

where H_s and H_e are the "free" Hamiltonians of $\mathcal{S}$ and $\mathcal{E}$, respectively, and H_{int} is some interaction term. Given the pre- and post-selected states, $|\psi_i(t_1)\rangle = |s_1\rangle \otimes |e_1\rangle$ and $|\psi_f(t_2)\rangle = |s_2\rangle \otimes |e_2\rangle$, the TS in the Schrödinger representation is $\hat{\varrho}_{s+e}(t) = U(t-t_1)|\psi_i\rangle\langle\psi_f|U^\dagger(t-t_2)$, where $U(t_2 - t_1) = \exp(-i \int_{t_1}^{t_2} H_{tot} dt')$. Limiting out observations only to the subsystem $\mathcal{S}$, we would like to compute the probabilities for observables of the form $A = A_s \otimes 1_e$, where A_s operates in the Hilbert space $\mathcal{H}_\mathcal{S}$ of $\mathcal{S}$ and 1_e is a unit operator in $\mathcal{H}_\mathcal{E}$.

This probability can be expressed in a simple form by Eq. (6), with $\hat{\varrho} = \hat{\varrho}_{s+e}(t)$ and $P_a = (P_a)_s \otimes I_e$. Obviously, since the projection operator acts only in $\mathcal{H}_{\mathcal{S}}$, we can trace over $\mathcal{E}$ and represent this probability in terms of a reduce TS $\hat{\varrho}_s$:

$$\mathcal{P}rob(a,t|s_2,e_2,s_1,e_1) = \frac{|\langle P_a, \hat{\varrho}_s(t)\rangle|^2}{\sum_{a'} |\langle P_{a'}, \hat{\varrho}_s(t)\rangle|^2} \tag{11}$$

where,

$$\hat{\varrho}_s = \hat{\varrho}_s(t; s_2, e_2, s_1, e_1) = \frac{1}{N}\mathrm{tr}_e \hat{\varrho}_{s+e} \tag{12}$$

The time independent normalization, $N = \langle e_2(t_2)| \exp[-iH_e(t_2 - t_1)]|e_1(t_1)\rangle$, was chosen for later convenience. At intermediate times $\mathcal{S}$ is completely described in terms of the reduced TS.

Notice that at the boundaries, $t = t_1$ and $t = t_2$, the reduced TS has a simple generic form:

$$\hat{\varrho}_s(t_2) = (\hat{U})_w|s_2\rangle\langle s_1| = |s'\rangle\langle s_1| \tag{13}$$

and

$$\hat{\varrho}_s(t_1) = |s_1\rangle\langle s_2|(\hat{U}^\dagger)_w = |s_2\rangle\langle s''| \tag{14}$$

where

$$(\hat{U})_w = \frac{\langle e_2|\hat{U}(t_2 - t_1)e^{-iH_e(t_2-t_1)}|e_1\rangle}{\langle e_2|e^{-iH_e(t_2-t_1)}|e_1\rangle} \tag{15}$$

is the 'weak value' [12] of the evolution operator $\hat{U}$ with respect to the 'free' environment's pre and post-selected states. Hence, $(\hat{U})_w$ is an operator in the Hilbert space $\mathcal{H}_{\mathcal{S}}$. On the other hand, due to the interaction with the environment, at intermediate times, $t \in (t_1, t_2)$, the reduced TS is generally a non-reducible "entangled" TS:

$$\hat{\varrho}_s(t) = \sum C_{s's''}(t)|s'\rangle\langle s''|. \tag{16}$$

This effect of "decoherence" and then "recoherence" of the reduced two-state, as expressed in Equations (13), (14), and (16), stands in the heart of this paper. The final post selection

of the environment "force's" the two-state to recohere at the final condition to a generic two-state.

The effect of post-selecting the environment exists even if the sub-system $\mathcal{S}$ is not post-selected, i.e. the condition at $t = t_2$ is imposed only on $\mathcal{E}$. In this case the probability to find $A = a$ at $t \in (t_2, t_1)$ is given by

$$\mathcal{P}rob(a, t|r_2, r_1, s_1) = \frac{\sum_{s_2} |\langle P_a, \hat{\varrho}_s(s_2)\rangle|^2}{\sum_{s_2,a'} |\langle P_{a'}, \hat{\varrho}_s(s_2)\rangle|^2} \tag{17}$$

The sum above in over all possible eigenstates, $\{|s_2\rangle\}$, of an *arbitrary* complete set of operator(s) $\hat{S}$. This probability is independent on the choice of $\hat{S}$.

Although, in this case, there is only one (initial) condition on $\mathcal{S}$, due to the interaction with the pre- and post-selected environment, the sub-system $\mathcal{S}$ can not in general be described in terms of a pure or a mixed density matrix. Equation (17) can be rewritten as

$$\mathcal{P}rob(a, t|r_2, r_1, s_1) = \frac{\mathrm{tr} P_a \rho(a)}{\sum_{a'} P_{a'} \rho(a)} \tag{18}$$

where

$$\rho(a) = \sum_{s_2} \hat{\varrho}(s_2) P_a \hat{\varrho}^\dagger(s_2) \tag{19}$$

The object ρ corresponds to a density matrix only if it is independent of a. Intuitively, this happens when the condition at $t = t_2$ on $\mathcal{E}$ does not "add" information. Let us examine this question more closely. When $t \to t_2$, we have $\hat{\varrho}_s \to (\hat{U})_w |s_1\rangle\langle s_1| = |s'\rangle\langle s_2|$ and by (19)

$$\rho(a, t_2) = \sum_{s_2} |s'\rangle\langle s_2|a\rangle\langle a|s_2\rangle\langle s'| = |s'\rangle\langle s'| \tag{20}$$

is independent of a. Therefore, near the final condition there is *always* an effective pure state. The initial state of the open system, $|s_1\rangle$ is mapped to a final pure state $|s'\rangle$ by the "weak evolution operator"

$$\hat{U}_w |s_1\rangle = |s'\rangle, \tag{21}$$

Near the initial condition, Eq. (19) yields

$$\rho(a,t_1) = (\langle a|(\hat{U})_w(\hat{U}^\dagger)_w|a\rangle)|s_1\rangle\langle s_1| = C(a)|s_1\rangle\langle s_1| \tag{22}$$

The effective density matrix is proportional to a pure state, but the probability at $t = t_1$ depends on an unconventional normalization

$$\mathcal{P}rob(a,t_1) = \frac{\mathrm{tr} P_a \rho(a,t_1)}{\sum_{a'} \rho(a',t_1)} = \frac{C(a)}{\sum_{a'} C(a')} |\langle a|s_1\rangle|^2 \tag{23}$$

Unless $\hat{U}_w \hat{U}_w^\dagger = 1$, this probability may depend on the nature of the final condition on the environment. For example, if without the post selection we would have $\mathcal{P}rob(a) = 1$ and $\mathcal{P}rob(b \neq a) = 0$, then these probability are not effected by the final post selection of $\mathcal{E}$. But the post selection of $\mathcal{E}$ does generally modify the probability in intermediat e cases as $0 < \mathcal{P}rob(c) < 1$.

At any intermediate times, $t_1 < t < t_2$, the effective density matrix (19) will be a-dependent, and hence a complete description in terms of a unique density matrix is not possible. It is interesting however, that for a a limited class of observables, whose nature depends on the coupling with the environment, we can still construct an effective density matrix. To see this, let us choose the (otherwise arbitrary) set $\{|s_2\rangle\}$ in Equation (17), a s eigenvalues of a complete set of an operators $\hat{S}_k$ that *commute* with H_{int}. In this case, for a given s_2 the TS has a generic form : $\hat{\varrho}(s_2, t) = |s', t\rangle\langle s_2, t|$. Therefore, for an operator $A = \sum aP_a$ which is conjugate to one of the operators $\hat{S}_k$, we have $\langle s_2|P_a|s_2\rangle = constant$. This implies that $\rho(a)$, the effective density matrix, does not depend on a. Hence, if one measures *only* this limited class of observables, one can use the effective density matrix given by $\rho_{den}(t) = \sum_{s_2} \hat{\varrho}\hat{\varrho}^\dagger$. This density matrix is pure near the conditions at $t = t_1$ and $t = t_2$, but generally corresponds to a mixed state at $t_2 > t > t_1$.

4 A simple example

To exemplify these ideas we now consider a solvable model, which was used to demonstrate decoherence [8], of a spin half particle (the system) coupled to N spin half particles (the environment). Setting the free part of the Hamiltonian to zero the interaction part is taken as

$$H_{int} = \sum_{k=1}^{N} g_k \sigma_z \sigma_z^{(k)} \tag{24}$$

In term of the eigenstates of σ_z and $\sigma_z^{(k)}$, the conditions can be expressed as

$$|\psi_1(t=0)\rangle = \Big(a|\uparrow\rangle + b|\downarrow\rangle\Big) \prod_k \Big(\alpha_k|\uparrow_k\rangle + \beta_k|\downarrow_k\rangle\Big) = |s_1\rangle|e_1\rangle \tag{25}$$

and

$$|\psi_2(t=T)\rangle = \Big(a'|\uparrow\rangle + b'|\downarrow\rangle\Big) \prod_k \Big(\alpha'_k|\uparrow_k\rangle + \beta'_k|\downarrow_k\rangle\Big) = |s_2\rangle|e_2\rangle \tag{26}$$

The reduced TS can be derived according to Eq. (12), by tracing over the $k = 1, ..N$ spins. The result is:

$$\begin{aligned}\hat{\varrho}_s(t) = \frac{1}{\chi(0)}\Big\{ & aa'^*\chi(T)|\uparrow\rangle\langle\uparrow| + bb'^*\chi(-T)|\downarrow\rangle\langle\downarrow| \\ & + ab'^*\chi(2t-T)|\uparrow\rangle\langle\downarrow| + ba'^*\chi(T-2t)|\downarrow\rangle\langle\uparrow|\Big\}\end{aligned} \tag{27}$$

where

$$\chi(t') = \prod_k \Big(\alpha_k \alpha_k'^* e^{ig_k t'} + \beta_k \beta_k'^* e^{-ig_k t'}\Big). \tag{28}$$

At the initial and final conditions, the TS reduces to

$$\hat{\varrho}_s(t=0) = |s_1\rangle\langle s_2|\hat{U}_w^\dagger(T) = \frac{1}{\chi(0)}\Big(a|\uparrow\rangle + b|\downarrow\rangle\Big) \otimes \Big(a'^*\chi(T)\langle\uparrow| + b'^*\chi(-T)\langle\downarrow|\Big) \tag{29}$$

and

$$\hat{\varrho}_s(t=T) = \hat{U}_w(T)|s_1\rangle\langle s_2| = \frac{1}{\chi(0)}\Big(a\chi(T)|\uparrow\rangle + b\chi(-T)|\downarrow\rangle\Big) \otimes \Big(a'^*\langle\uparrow| + b'^*\langle\downarrow|\Big) \tag{30}$$

where the 'weak evolution operator' is

$$\hat{U}_w(t) = \frac{\chi(\sigma_z t)}{\chi(0)} \tag{31}$$

At intermediate times $\hat{\varrho}_s(t)$ can not generally be reduce to a generic TS.

Let us examine the case that only the N spins (environment) are post selected. In this case we need to use equation (17) and sum over all the final possibilities. Obviously, it is most convenient to sum over final eigenstates of σ_z. Hence we have two possible two-states:

$$\hat{\varrho}_s(t,\uparrow) = \frac{1}{\chi(0)}\Big(a\chi(T)|\uparrow\rangle + b\chi(T-2t)|\downarrow\rangle\Big) \otimes \langle\uparrow| \tag{32}$$

and

$$\hat{\varrho}_s(t,\downarrow) = \frac{1}{\chi(0)}\Big(a\chi(2t-T)|\uparrow\rangle + b\chi(-T)|\downarrow\rangle\Big) \otimes \langle\downarrow| \tag{33}$$

The effective density matrix (19) is in this case $\rho(a) = \hat{\varrho}_s(\uparrow)P_a\hat{\varrho}_s^\dagger(\uparrow) + \hat{\varrho}_s(\downarrow)P_a\hat{\varrho}_s^\dagger(\downarrow)$. Clearly, if we measure only σ_x or σ_y this expression reduces to $\rho_{eff} = \frac{1}{2}\Big(\hat{\varrho}_s(\uparrow)\hat{\varrho}_s^\dagger(\uparrow) + \hat{\varrho}_s(\downarrow)\hat{\varrho}_s^\dagger(\downarrow)\Big)$. Therefore, for these observables we have an effective density matrix:

$$\begin{aligned}\rho_{eff} = \frac{1}{2|\chi(0)|^2}\Big[\ & |a|^2\Big(|\chi(T)|^2 + |\chi(2t-T)|^2\Big)|\uparrow\rangle\langle\uparrow| \\ & +|b|^2\Big(|\chi(-T)|^2 + |\chi(T-2t)|^2\Big)|\downarrow\rangle\langle\downarrow| \\ & +ab^*\Big(\chi(T)\chi^*(T-2t) + \chi(2t-T)\chi^*(-T)\Big)|\uparrow\rangle\langle\downarrow| \\ & + a^*b\Big(\chi(T-2t)\chi^*(T) + \chi(-T)\chi(2t-T)\Big)|\downarrow\rangle\langle\uparrow|\ \Big]\end{aligned} \tag{34}$$

At the boundaries this expression reduces to

$$\rho_{eff}(t=T) = \frac{1}{|\chi(0)|^2}\Big(a\chi(T)|\uparrow\rangle + b\chi(-T)|\downarrow\rangle\Big) \otimes \Big(a\chi^*(T)\langle\uparrow| + b\chi^*(-T)\langle\downarrow|\Big) \tag{35}$$

and

$$\rho_{eff}(t=0) = \frac{1}{2|\chi(0)|^2}\Big[|\chi(T)|^2 + |\chi(-T)|^2\Big]\Big(a|\uparrow\rangle + b|\downarrow\rangle\Big)\otimes\Big(a^*\langle\uparrow| + b^*\langle\downarrow|\Big) \tag{36}$$

The initial and final effective density matrix corresponds to a pure state. However notice that the norm of the initial and final pure state is not the same. This reflects the non-unitarity of the 'weak evolution operator'. Hence, for a limited set of observables, we obtained a description in terms of a density matrix which initially decoheres and finally recoheres back to a pure state. It is now amusing to note that by fixing the initial and final states of the N spins to satisfy: $|\alpha_k \alpha_k'^*|^2 = |\beta_k \beta_k'^*|^2 = 1/2$, (e.g. pre and post selection of $\sigma_x^{(k)} = 1, \quad (k = 1, .., N)$), we can arrange that near the in itial condition, the state of the system is described for *any* observable by a pure state. In this case the system in intermediate time is (effectively), for some observables, in a mixed state, while for other observables, even a mixed state not exits. The system always 'recoheres' back to a pure state.

5 Reduced two-state dynamics

The two-state of a closed system satisfies a Liouville Equation. By focusing on a subsystem, and tracing over the environment's degrees of freedom we will also modify the equation of motion of the the reduced two-state. Some additional terms are now necessary to accommodate for the effect of the 'external' e nvironment. This problem is reminiscent to the well studied issue of environment induced decoherence. There is however a significant difference between the two problems. As we have seen, when the conditions correspond to pure states, the *exact* solution for the TS must be of generic (direct product) form, *both*, initially at $t = t_1$ and finall y at $t = t_2$. Therefore, the resulting dynamical equation must have the non-trivial property that given any two conditions for $\mathcal{S}$, it evolves an initially generic TS to an "entangled TS" at intermediate times, and back to a generic TS at at the final condition. Such a 'fine tuning' requires cushion when approximations are used to derive the corrections to the Liouville Equation. For example, in deriving the equation

of motion to the reduced density matrix, it is usually assumed that one can use the 'non-reversible' approximation that the density matrix can be factorized to a product of two density matrix of form: $\rho_{density} = \rho_s \times \rho_e$. This simplifies considerably the computations. However, in our case such an approximation is invalidated since the TS can not be factorized in such a way at any time. In fact a naive usage of such a factorization leads to an equation of motion with no solutions for the two boundary condition problem.

In the following we shall derive perturbatively the modified Liouville Equation. Therefore we expect our solution to be valid only in the weak coupling regime $\lambda T < 1$, where λ is the coupling constant ($H_{int} = \lambda H_I$), and $T = t_2 - t_1$. For simplicity we shall assume a time independent Hamiltonian and that H_{int} is an analytic function. In the following, it will be most convenient to use the interaction representation. Setting $t_1 = 0$ and $t_2 = T$ we define the TS in the interaction representation as

$$\hat{\varrho}_{int}(t) = e^{iH_0 t}\hat{\varrho}(t)e^{-iH_0 t} \tag{37}$$

The equation of motion of the closed system is

$$\partial_t \hat{\varrho}_{int} = -i\Big[[H_{int}]_I, \hat{\varrho}_{int}\Big] \tag{38}$$

where $[\mathcal{O}]_I \equiv e^{iH_0 t}[\mathcal{O}]e^{-iH_0 t}$.

Now define $\hat{\varrho}_0 = |\psi_1(0)\rangle\langle\psi_2(T)|\exp(-iH_0 T) = \hat{\varrho}_{s0} \otimes \hat{\varrho}_{e0}$, which is the free ($H_{int} = 0$) two-state at $t = 0$. In terms of $\hat{\varrho}_0$ we have

$$\hat{\varrho}_{int}(t) = \Big[e^{-iHt}e^{+iH_0 t}\Big]_I \hat{\varrho}_0 \Big[e^{-iH_0(t-T)}e^{+iH(t-T)}\Big]_I \tag{39}$$

For simplicity let us assume that $[H_0, H_{int}] = 0$, hence

$$\hat{\varrho}_{int}(t) = e^{-i[H_{int}]_I t}\hat{\varrho}_0 e^{+i[H_{int}]_I (t-T)} \tag{40}$$

Although the exact solution $\hat{\varrho}_{int}$ can not be factorized, we can use (40) to expend it in

powers of $\hat{\varrho}_0 = \hat{\varrho}_{s0} \times \hat{\varrho}_{e0}$. Putting from now on $\hat{\varrho} = \hat{\varrho}_{int}$ and $\lambda H_I = [H_{int}]_I$, we have:

$$\hat{\varrho}(t) = \hat{\varrho}_0 - i\lambda t H_I \hat{\varrho}_0 - i\lambda(T-t)\hat{\varrho}_0 H_I + O(\lambda^2) \tag{41}$$

The free TS $\hat{\varrho}_0$ is factorizable, and we can now trace over $\mathcal{E}$. Therefore,

$$\hat{\varrho}_s(t) \equiv \frac{\mathrm{tr}_e \hat{\varrho}}{\mathrm{tr}_e \hat{\varrho}_{e0}} = \hat{\varrho}_{s0} - i\lambda t (H_I)_w \hat{\varrho}_{s0} - i\lambda(T-t)\hat{\varrho}_{s0}(H_I)_w + O(\lambda^2) \tag{42}$$

where $(...)_w$ stands for the weak value with respect to free environment's two-state, and is defined by $\mathcal{O}_w = \mathrm{tr}\mathcal{O}\hat{\varrho}_{e0}/\mathrm{tr}\hat{\varrho}_{e0}$. The last expression can be also inverted to

$$\hat{\varrho}_{s0} = \hat{\varrho}_s(t) + i\lambda t (H_I)_w \hat{\varrho}_s(t) + i\lambda(T-t)\hat{\varrho}_s(t)(H_I)_w + O(\lambda^2) \tag{43}$$

Substituting (41) into the Liouville equation and tracing over $\mathcal{E}$ yields

$$\partial_t \hat{\varrho}_s(t) = -i\lambda[(H_I)_w, \hat{\varrho}_{s0}] - \lambda^2\Big[\Big(H_I, tH_I\hat{\varrho}_{s0} + (T-t)\hat{\varrho}_{s0}H_I\Big)_w\Big] + O(\lambda^3) \tag{44}$$

Finally, we can use (43) to reexpress the last equation in terms of $\hat{\varrho}(t)$. We get

$$\partial_t \hat{\varrho}_s(t) = -i\lambda[(H_I)_w, \hat{\varrho}_s(t)] \tag{45}$$

$$-\lambda^2\Big[\Big(H_I, tH_I\hat{\varrho}_s(t) + (T-t)\hat{\varrho}_s(t)H_I\Big)_w\Big] + \Big[(H_I)_w, \Big(tH_I\hat{\varrho}_s(t) + (T-t)\hat{\varrho}_s(t)H_I\Big)_w\Big] + O(\lambda^3) \tag{46}$$

Let us consider some examples. For a generic interaction of the form:

$$H_I = \lambda Q_i L_i \tag{47}$$

where the Q_i's are some system variables and L_i reservoir variables, we get in the free case $(H_s = H_e = 0)$:

$$\partial_t \hat{\varrho}_s(t) = -i\lambda(L_i)_w[Q_i, \hat{\varrho}_s] - \lambda^2\Delta_{ij}[Q_i,\ tQ_j\hat{\varrho}_s + (T-t)\hat{\varrho}_s Q_j] + O(\lambda^3) \tag{48}$$

where

$$\Delta_{ij} = (L_i L_j)_w - (L_i)_w (L_j)_w \tag{49}$$

Typically, the first order is the Liouville equation with a "weak" Hamiltonian". The second order corrections, are proportional to the "weak uncertainty" Δ_{ij}. Higher order may be easily computed, but become very cumbersome. It is straightforward to rewrite (48) to the case that L_i and Q_i are not constants of motion, or to any other polynomial interaction.

Simplifying the interaction even further, we set $Q_1 = \sigma_z$, $L_1 = L_z \equiv L$ and $L_i = Q_i = 0$ for $i \neq 1$. This corresponds to a spin half subsystem which interacts with the z component of the angular momentum of the environment. Equation (48) reduces to

$$\partial_t \hat{\varrho}_s = -i\lambda L_w[\sigma_z, \hat{\varrho}_s] - 2\lambda^2 \Delta L_w (2t - T)(\hat{\varrho}_s - \sigma_z \hat{\varrho}_s \sigma_z) + O(\lambda^3) \tag{50}$$

where $\Delta L_w = (L^2)_w - (L_w)^2$.

We can easily verify that for every two initial and final conditions for $\mathcal{S}$, there exists an appropriate solution. It is only the second order term that can induce transition from generic to non-generic (entangled) two-state. In terms of the notation $\hat{\varrho}_{\uparrow\downarrow} = |\uparrow\rangle\langle\downarrow|$, etc, the general solution of Eq. (50) is

$$\hat{\varrho}_{\uparrow\uparrow}(t) = \hat{\varrho}_{\uparrow\uparrow 0}, \quad \hat{\varrho}_{\downarrow\downarrow}(t) = \hat{\varrho}_{\downarrow\downarrow 0} \tag{51}$$

$$\hat{\varrho}_{\uparrow\downarrow}(t) = \exp\Big[-i2\lambda L_w t - 4\lambda^2 \Delta L_w (t^2 - Tt)\Big] \hat{\varrho}_{\uparrow\downarrow 0} \tag{52}$$

$$\hat{\varrho}_{\downarrow\uparrow}(t) = \exp\Big[+i2\lambda L_w t - 4\lambda^2 \Delta L_w (t^2 - Tt)\Big] \hat{\varrho}_{\downarrow\uparrow 0} \tag{53}$$

Clearly, due to the factor $t^2 - Tt$, the second order contributions vanishes on at the conditions. By substituting $\lambda L = \sum g_k \sigma_z^k$, it can be verified that this solution agrees up to corrections of order $O(\lambda^3)$ with the exact solution given by equation (27).

Due to the continues interaction with each of the spins in the latter problem, the validity of Equation (50) is limited by the constraint $T < 1/\lambda$. We shall now compare this system to the other extreme case, in which the subsystem interacts with each of the particles of

the environment separately, and only for a very short time $\Delta t = \tau$, such that $\tau\lambda << 1$. In this way the weak coupling condition is satisfied, and our modified Liouville Eq. can be applied also for long tim es. Let the environment be composed of N non-interacting particles. The interaction Hamiltonian for this case is given by [7]

$$H_I = \lambda \sum_{n=1}^{N} f_n(t) H_n \tag{54}$$

where $f_n(t) = \theta(t - n\tau) - \theta(t - (n+1)\tau)$ with $\theta(t)$ as the step function is nonzero only for $t \in (n\tau, (n+1)\tau)$. H_n is the interaction of $\mathcal{S}$ with the nth particle. Let us further assume that H_n can be regarded as (or i s) constant during the interaction times τ. For $H_n = \sigma_z L_{nz} = \sigma L_n$ we get

$$\partial_t \hat{\varrho}_s = -i\lambda \sum_n f_n(t) L_{nw}[\sigma, \hat{\varrho}_s] - \lambda^2 \sum_n f_n \Delta_{nn}(2t - (2n+1)\tau)(\hat{\varrho}_s - \sigma\hat{\varrho}_s\sigma)$$
$$- \lambda^2 \sum_{n,m=1}^{m=n-1} f_n \Delta_{nm}(n\tau)[\sigma, \sigma\hat{\varrho}_s] - \lambda^2 \sum_{n,m=n+1}^{m=N} f_n \Delta_{nm}(N-n-1)\tau[\sigma, \hat{\varrho}_s\sigma] \tag{55}$$

where $\Delta_{nm} = (L_n L_m)_w - (L_n)_w (L_n)_w$. If the initial and final states of the environment are given by a product state, $\prod_k \otimes|e_k\rangle$ of the N particles, there are no correlations between the weak values different particles in the reservoir and $\Delta_{nm} = ((L_n^2)_w - (L_{nw})^2)\delta_{nm}$. Therefore, in this case the two last terms on the right hand side of equation (55) vanish. Integrating (55) we see that after each "step", when the interaction with the n'th particle in the environment is completed, the accumulated contribution of the second term drops to zero. The TS remains 'pure' up to fluctuations of order $O(\lambda^2\tau^2)$. In this sense, we can say that the post-selection of the environment prevents decoherence of the subsystem.

t a time scale T.

Acknowledgment

I would like to thank Bill Unruh for helpful discussions.

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